6.3
Trigonometric Equations

Learning objectives:

1. To find principle solution and general solution of a trigonometric equation.
2. To use different methods to solve trigonometric equations.
   And
3. To practice the related problems.

Solving Trigonometric Equations

Trigonometric equations are equations involving trigonometric functions of unknown angles and they, unlike identities, are satisfied only by particular values of the unknown angles. For example, $\sin x \,.\, \csc x = 1$ is an identity, being satisfied by every value of x for which $\sin x$ and $\csc x$ are defined.

$\sin x = 0$; it is not satisfied by $x = \frac{\pi}{4}$ or $\frac{\pi}{2}$. Since it is not satisfied by every value of x for which it is defined, it is not an identity. It is an equation and we will find the particular values of x for which this equation is satisfied.

A ***solution*** of a trigonometric equation is a value of the angle x which satisfies the equation.

If a given equation has one solution, it has in general an unlimited number of solutions due to the periodicity of the trigonometric functions.

Two solutions of $\sin x = 0$ are $x = 0$ and $x = \pi$.

The complete solution of $\sin x = 0$ is given by

$$x = 0 + 2n\pi, \qquad x = \pi + 2n\pi$$

where n is any integer. Both these expressions can be combined into a single expression $x = n\pi$, where n is any integer.

The solution consisting of all possible solutions of a trigonometric equation is called its ***general solution***.

There is no general method for solving trigonometric equations. Several standard procedures are employed in the solution of trigonometric equations.

The numerically least angle of the solution is called the ***principal value or principle solution***.

For example, find the principal value of $\sin x = \frac{1}{2}$.

The numerically least value will be in the first quadrant. Therefore, the principal value is $x = \frac{\pi}{6}$

- Find the principal value of x satisfying $\sin x = -\frac{1}{2}$.

The sine is negative in 3rd or 4th quadrant. Therefore, the principal value is $x = -\frac{\pi}{6}$

- Find the principal value of x satisfying $\tan x = -1$.

Tan is negative in 2^{nd} and 4^{th} quadrants. The principal value is $x = -\frac{\pi}{4}$

- Find the principal value of x satisfying $\cos x = \frac{1}{2}$.

Cosine is positive in first and fourth quadrants. The principal value is $x = \frac{\pi}{3}$

Principal value lies in the first quadrant. It is never numerically greater than π. The clockwise or anticlockwise direction is chosen depending on whether the angle is in 3rd and 4th quadrant or in first and second quadrants.

Factorable Equations

Solve $\sin x - 2\sin x\, cos\, x = 0.$

Factoring,

$$\sin x - 2\sin x \cos x = sinx(1 - 2cosx) = 0$$

Setting each factor equal to zero, we get

$$\sin x = 0 \Longrightarrow x = 0, \pi$$

$$1 - 2\cos x = 0 \Longrightarrow \cos x = \frac{1}{2} \Longrightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Expressible in terms of a Single Function

Solve $2\, tan^2 x + sec^2 x = 2$

$2\, tan^2 x + sec^2 x = 2$

$2\, tan^2 x + 1 + tan^2 x = 2 \Longrightarrow\ 3tan^2 x = 1$

$\Longrightarrow\ \tan x = \pm \frac{1}{\sqrt{3}}$

For $\tan x = \frac{1}{\sqrt{3}}$, $x = \frac{\pi}{6}, \frac{7\pi}{6}$

For $\tan x = -\frac{1}{\sqrt{3}}$, $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

Solve $\sec x + \tan x = 0$

$$\frac{1}{cos\ x} + \frac{\sin x}{cos\ x} = 0$$

Multiplying by $\cos x$, we have $1 + \sin x = 0,\ \sin x = -1.$
Then $x = \frac{3\pi}{2}$

However, neither $\sec x$ nor $\tan x$ is defined when $x = \frac{3\pi}{2}$ and the equation has no solution.

This illustrates that there is a need to check the solution before accepting it as a solution of the equation.

Squaring Both Members of the Equation

Solve $\sin x + \cos x = 1$

We write the equation in the form $\sin x = 1 - \cos x$ and square both members. We have

$$sin^2 x = 1 - 2\cos x + cos^2 x$$

$\Longrightarrow 1 - cos^2 x = 1 - 2\cos x + cos^2 x$

$\Longrightarrow 2\, cos^2 x - 2\cos x = 0 \Longrightarrow\ 2\cos x(\cos x - 1) = 0$

From $2cos\ x = 0$, $x = \frac{\pi}{2}, \frac{3\pi}{2}$

From $\cos x = 1$, $x = 0$

Check:

For $x = 0$, $\sin x + cosx = 0 + 1 = 1$

For $x = \frac{\pi}{2}$, $\sin x + cosx = 1 + 0 = 1$

For $x = \frac{3\pi}{2}$, $\sin x + cosx = -1 + 0 \neq 1$

Thus, the solution is $x = 0$ and $\frac{\pi}{2}$.

The value $x = \frac{3\pi}{2}$, called an *extraneous solution*, was introduced by squaring the members.

General solution of the equations

$\sin x = 0, \cos x = 0$ and $\tan x = 0$

(i). If $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then $sin\theta = 0$ if and only if $\theta = 0$. Thus the principal solution of $sin\, x = 0$ is 0. Let $\theta \in R$ be any solution of $sin\, x = 0$. Then these exists a $k \in Z$ such that

$$k \leq \frac{\theta}{2\pi} < k + 1$$

$$\Rightarrow 2\pi k \leq \theta < 2\pi k + 2\pi$$

$$i.e., 0 \leq \theta - 2k\pi < 2\pi$$

Since θ and $\theta - 2\pi$ are co terminal angles,

$$\sin\theta = \sin(\theta - 2k\pi) = 0$$

$\Rightarrow\ \theta - 2k\pi = 0$ or $\pi \Rightarrow \theta = 2k\pi$ or $(2k+1)\pi, k \in Z$

i.e., $\theta = n\pi, n \in Z$

Thus,$sin\theta = 0 \ \Leftrightarrow \theta = n\pi, n \in Z$

Therefore, the general solution of the equation

$$\boldsymbol{sinx = 0\ is\ x = n\pi, n \in Z}$$

(ii).The principal solution of $cosx = 0$ is $x = \frac{\pi}{2}$.Further,

$$cos\, x = 0 \ \Leftrightarrow \sin\left(x - \frac{\pi}{2}\right) = 0 \Leftrightarrow x - \frac{\pi}{2} = n\pi,\ n \in Z$$

$$\Leftrightarrow x = n\pi + \frac{\pi}{2}$$

$$\Leftrightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$$

Thus, the general solution of $cosx = 0$ is

$$\boldsymbol{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

(iii).The principal value of $tan\, x = 0$ is $x = 0$. Further,

$$tanx = 0 \Leftrightarrow sinx = 0 \ \Leftrightarrow x = n\pi, n \in Z$$

Thus, the general solution of $tanx = 0$ is

$$\boldsymbol{x = n\pi, \qquad n \in Z}$$

NOTE: The general solution of $cot\, x = 0$ is given by

$$\boldsymbol{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

General solution of $sinx = k, |k| \leq 1$

Since $|k| \leq 1$, there exists a principal solution say α,

i.e., There is a $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $sin\,\alpha = k$

Let θ be any solution of $sin\,x = k$, then

$$sin\theta = sin\alpha \Leftrightarrow sin\theta - sin\alpha = 0$$

$$\Leftrightarrow 2cos\frac{\theta+\alpha}{2}.sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow 2cos\frac{\theta+\alpha}{2} = 0 \text{ or } sin\frac{\theta-\alpha}{2} = 0$$

Now, $cos\frac{\theta+\alpha}{2} = 0 \Leftrightarrow \frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}, n \in Z$

$$\Leftrightarrow \theta = (2n+1)\pi - \alpha, n \in Z$$

and, $sin\frac{\theta-\alpha}{2} = 0 \Leftrightarrow \frac{\theta-\alpha}{2} = n\pi, n \in Z$

$$\Leftrightarrow \theta = 2n\pi + \alpha, n \in Z$$

Combining those two, $\theta = n\pi + (-1)^n\alpha, n \in Z$

Thus, the general solution of the equation $sinx = k$, $|k| \leq 1$ is

$$\boldsymbol{x = n\pi + (-1)^n\alpha}$$

Where α is the principal solution (or any solution) of the equation.

By similar argument we prove the following

- The general solution of the equation of $cosx = k, |k| \leq 1$ is $\boldsymbol{x = 2n\pi \pm \alpha, n \in Z}$

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where α is the principal solution (or any solution) of the equation

- The general solution of the equation $tanx = k, \quad k \in R$ is $\boldsymbol{x = n\pi + \alpha,\ n \in Z}$

where α is the principal solution (or any solution) of the equation.

Summary

The equation $f(x) = k$	Range of k	The interval in which the principal solution α lies	General solution
$\sin x = k$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$n\pi + (-1)^n\alpha, n\epsilon Z$
$\cos x = k$	$[-1, 1]$	$[0, \pi]$	$2n\pi \pm \alpha, n\epsilon Z$
$\tan x = k$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$n\pi + \alpha, n\epsilon Z$
$\csc x = k$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	$n\pi + (-1)^n\alpha, n\epsilon Z$
$\sec x = k$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	$2n\pi \pm \alpha, n\epsilon Z$
$\cot x = k$	R	$(0, \pi)$	$n\pi + \alpha, n\epsilon Z$

Example:

Solve the equation $\sin x + \sin 5x = \sin 3x$.

$$\Rightarrow 2\sin 3x \cos 2x = \sin 3x$$

$$\Rightarrow \sin 3x\,(2\cos 2x - 1) = 0$$

Therefore, $\sin 3x = 0$, or $\cos 2x = \frac{1}{2}$

If $\sin 3x = 0$, then $3x = n\pi, n\epsilon z$.

If $\cos 2x = \frac{1}{2}$, then $2x = 2n\pi \pm \frac{\pi}{3}, n\epsilon z$

Hence $x = \frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{6}, n\epsilon z$.

The general solution of $\boldsymbol{sin^2x = k, 0 \leq k \leq 1}$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$**, where $\boldsymbol{\alpha}$ is a solution of $\boldsymbol{sin^2x = k}$**

Proof:

The trigonometric equation $sinx = k$ has a solution if and only if $k \in [0,1]$. Thus there exists a solution say $\alpha \in R$ such that $sin^2x = sin^2\alpha$. Now

$$sin^2x = sin^2\alpha \Leftrightarrow 1 - 2sin^2x = 1 - 2sin^2\alpha$$

$$\Leftrightarrow cos2x = cos2\alpha.$$

$$\Leftrightarrow 2x = 2n\pi \pm 2\alpha,\ n \in Z$$

$$\Leftrightarrow \boldsymbol{x = n\pi \pm \alpha,\ n \in Z}$$

where α is a solution of $sin^2x = k$.

By a similar method we prove the following

- The general solution of $cos^2x = k, 0 \leq k \leq 1$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$, where α is a solution of $cos^2x = k$

- The general solution of $tan^2x = k, 0 \leq k < \infty$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$, where α is a solution of $tan^2x = k$

Equations of the form $a\cos\theta + b\sin\theta = c$

We divide both sides of the equation by $\sqrt{a^2+b^2}$, so that it may be written as

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

If we introduce the angle α, so that $\tan\alpha = \frac{b}{a}$

Then $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ $\quad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

Also, we introduce the angle β, so that

$$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$$

The equation can then be written

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \cos\beta$$

The equation is then $\cos(\theta-\alpha) = \cos\beta$.

The solution of this is $\theta - \alpha = 2n\pi \pm \beta$, so that

$$\boldsymbol{\theta = 2n\pi + \alpha \pm \beta}$$

where n is any integer.

Angles, such as α and β, which are introduced to facilitate computation are called **Subsidiary Angles**.

Example: Solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$

We have $\sqrt{a^2+b^2} = \sqrt{1+3} = 2$. We therefore write the equations as

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

Therefore

$$\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4} \Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

Taking the positive sign,

$$x = 2n\pi + \frac{\pi}{6} + \frac{\pi}{4} = 2n\pi + \frac{5\pi}{12}$$

Taking the negative sign,

$$x = 2n\pi + \frac{\pi}{6} - \frac{\pi}{4} = 2n\pi - \frac{\pi}{12}$$

P1.

Find the general solution of $2cos^2x\ tanx = tanx$.

Solution:

Given, $2cos^2x\ tanx = tanx$

$\Longrightarrow tanx(2cos^2x - 1) = 0 \Longrightarrow tanx = 0\ or\ (2cos^2x - 1) = 0$

$\Longrightarrow tanx = 0\ or\ cosx = \frac{1}{\sqrt{2}} \Longrightarrow x = 0, \pi\ or\ x = \frac{\pi}{4}$

$\Longrightarrow x = n\pi\ or\ x = n\pi \pm \frac{\pi}{4}, n \in Z$

P2.

Solve $3\cos 2\theta + 2 = 7\sin\theta$

Solution:

$$3\cos 2\theta + 2 = 7\ \sin\theta$$

$$\Longrightarrow 3\,(1 - 2sin^2\theta) + 2 = 7\,sin\theta$$

$$\Longrightarrow\ 6\,sin^2\theta + 7\,sin\theta - 5 = 0$$

$$\Longrightarrow (2\sin\theta\ - 1)(3\sin\theta + 5) = 0$$

$$\Longrightarrow\ \sin\theta\ = \frac{1}{2}, -\frac{5}{3}\ \text{and}\ \ \sin\theta\ = -\frac{5}{3} \notin [-1, 1]$$

$$\therefore\ \sin\theta\ = \frac{1}{2} \Longrightarrow\ \theta\ =\ \frac{\pi}{6}\ \text{(Principal solution)}$$

The general solution is $n\pi + (-1)^n\ \frac{\pi}{6}\ , n \in Z$

P3.

Solve: $\sqrt{3}\cos x - sinx = 1$

Solution:

Given, $\sqrt{3}\cos x - sinx = 1$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}sinx = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}sinx = \cos\frac{\pi}{3}$$

$$\Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}$$

P4.

If θ_1, θ_2 are solutions of the equation

$acos\theta + bsin\theta = c, tan\theta_1 \neq tan\theta_2$ and $a + c \neq 0,$ then find

the values of

i. $tan\theta_1 + tan\theta_2$

ii. $tan\theta_1 \cdot tan\theta_2$

Solution:

Given, $acos\theta + bsin\theta = c\ , a + c \neq 0$

$$\Rightarrow a\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right) + b\left(\frac{2tan\theta}{1+tan^2\theta}\right) = c$$

$$\Rightarrow a - atan^2\theta + 2btan\theta = c + ctan^2\theta$$

$$\Rightarrow (a+c)tan^2\theta - 2btan\theta + c - a = 0$$

$$\Rightarrow (a+c)tan^2\theta - 2btan\theta + c - a = 0$$

This is a quadratic equation in $tan\ \theta$. Since θ_1, θ_2 are roots of the given equation, we get $tan\theta_1$ and $tan\theta_2$ are roots of the equation.

$\therefore$ sum of the roots $tan\theta_1 + tan\theta_2 = \frac{2b}{a+c}$ and

Product of the roots $tan\theta_1 \cdot tan\theta_2 = \frac{c-a}{a+c}$

IP1.

Find the general solution of $2cos^2u = 1 - cosu$.

Solution:

Given, $2cos^2u = 1 - cosu$

$\Longrightarrow 2cos^2u + cosu - 1 = 0$

$\Longrightarrow (2cosu - 1)(cosu + 1) = 0$

$\Longrightarrow 2cosu - 1 = 0$ or $cosu + 1 = 0$

$\Longrightarrow cosu = \frac{1}{2}$ or $cosu = -1$

$\Longrightarrow u = \frac{\pi}{3}$ or $u = \pi$

General solution is $\left\{2n\pi \pm \frac{\pi}{3} \mid n \in Z\right\} \cup \{2n\pi \pm \pi \mid n \in Z\}$

IP2.

Solve: $2cos^2\theta + 11\ sin\theta = 7$

Solution:

$$2cos^2\theta + 11\ sin\theta = 7$$

$$\Longrightarrow 2\ (1 - sin^2\theta) + 11\ \ sin\theta - 7 = 0$$

$$\Longrightarrow\ 2 - 2\ sin^2\theta + 11\ sin\theta - 7 = 0$$

$$\Longrightarrow 2\ sin^2\theta - 11\ sin\theta + 5 = 0$$

$$\Longrightarrow (2\sin\theta - 1)(\sin\theta - 5) = 0$$

$$\Longrightarrow 2\sin\theta - 1 = 0\ \ or\ \ sin\theta - 5 = 0$$

$$\Longrightarrow\ \sin\theta = \frac{1}{2}\ \text{or}\ sin\theta = 5 > 1$$

$$\Longrightarrow\ \sin\theta = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{6}\ \ (Principle\ solutions)$$

The general solution is, $\theta = n\pi + (-1)^n\ \frac{\pi}{6}\ ,\ \ n \in Z$

IP3.

Solve: $2cosx + 2sinx = \sqrt{6}$

Solution:

Given, $2cosx + 2sinx = \sqrt{6}$

$$\Longrightarrow \frac{cosx}{\sqrt{2}} + \frac{sinx}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Longrightarrow \cos\frac{\pi}{4} cosx + sin\frac{\pi}{4} sinx = cos\frac{\pi}{6}$$

$$\Longrightarrow \cos\left(x - \frac{\pi}{4}\right) = cos\frac{\pi}{6}$$

$$\Longrightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6}$$

$$\Longrightarrow x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{6}$$

$$\therefore\ x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{6}$$

IP4.

If $\alpha\ and\ \beta$ are the solutions of $atan\theta + bsec\theta = c$, then show that $\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$

Solution:

Given, $a\ tan\theta + b\ sec\theta = c$

$$bsec\theta = c - atan\theta$$

$$\Longrightarrow b^2sec^2\theta = c^2 + a^2tan^2\theta - 2actan\theta$$

$$\Longrightarrow b^2(1 + tan^2\theta) = c^2 + a^2tan^2\theta - 2ac\ tan\theta$$

$$\Longrightarrow (a^2 - b^2)tan^2\theta - 2ac\ tan\theta + (c^2 - b^2) = 0$$

Also given, α and β are the solutions of θ.

$tan\alpha + tan\beta = \frac{2ac}{a^2-c^2}$ and $tan\alpha \cdot tan\beta = \frac{c^2-b^2}{a^2-b^2}$

$$\tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\ \cdot tan\beta} = \frac{\left(\frac{2ac}{a^2-c^2}\right)}{1-\left(\frac{c^2-b^2}{a^2-b^2}\right)} = \frac{2ac}{a^2-c^2}$$

1. Solve the following equations.

a) $sin\theta + sin7\theta = sin\ 4\theta$

b) $cos\theta + cos7\theta = cos4\theta$

c) $cos\theta + cos3\theta = 2\cos 2\theta$

d) $\sin 4\theta - \sin 2\theta = \cos 3\theta$

e) $cos\theta - sin3\theta = \cos 2\theta$

f) $\sin 7\theta = \sin\theta + \sin 3\theta$

g) $\cos\theta + \cos 3\theta = 0$

h) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

i) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$

j) $\cos n\theta = \cos(n-2)\theta + \sin\theta$

k) $\sin\frac{n+1}{2}\theta = \sin\frac{n-1}{2}\theta + \sin\theta$

l) $\sin m\theta + \sin n\theta = 0$

m) $\cos m\theta + \cos n\theta = 0$

n) $\sin 3\theta + \cos 2\theta = 0$

o) $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

p) $\sin\theta + \cos\theta = \sqrt{2}$

q) $sin^2\ n\theta - sin^2\ (n-1)\theta = sin^2\ \theta$